Appendix C Standard Errors for Non-Linear Models Using the Standard Normal Distribution

Marginal effects and their standard errors of interaction terms in non-linear models are not correctly estimated by standard statistical packages as pointed out by Ai and Norton (2001). We derive here the expressions for the standard errors of marginal effects calculated using a probit model.

Let $E[y|x_1, x_2, X] = \Phi(\beta_1 x_1 + \beta_2 x_2 + \beta_{12} x_1 x_2 + X\beta) = \Phi(I)$ represent the decision of interest, where an interaction term between the variables x_1 and x_2 is introduced. I represents the index value underlying the decision. X is a $1 \times K$ vector of remaining regressors and β the associated parameters vector.

The marginal effects of interest are:

$$\begin{aligned}
\mu_1 &= \frac{\partial \Phi(I)}{\partial x_1} = a\Phi'(I) \\
\mu_2 &= \frac{\partial \Phi(I)}{\partial x_2} = b\Phi'(I) \\
\mu_{12} &= \frac{\partial^2 \Phi(I)}{\partial x_1 \partial x_2} = \Phi'(I)\beta_{12} + ab\Phi''(I)
\end{aligned}$$

where $a = (\beta_1 + \beta_{12} x_2)$ and $b = (\beta_2 + \beta_{12} x_1)$.

The standard errors of μ_1, μ_2 and μ_{12} can be estimated by the Delta method. The later states that the variance of a parameter α_1 of the non-linear function $F(Z, \alpha)$ can be calculated by :

$$\sigma^2_{\alpha_1} = \frac{\partial}{\partial \alpha'}\left[\frac{\triangle F(Z,\alpha)}{\triangle z_1}\right] \Omega_\alpha \frac{\partial}{\partial \alpha}\left[\frac{\triangle F(Z,\alpha)}{\triangle z_1}\right]$$

where Ω_α is the covariance matrix of α. $\sigma^2_{\alpha_1}$ can be consistently estimated given that $\hat{\alpha}$ and $\hat{\Omega}_\alpha$ are consistent estimators of α and Ω_α.

Then, the variances of μ_1, μ_2 and μ_{12} are:

$$\sigma^2_{\mu_1} = \begin{bmatrix} a\Phi'' x_1 + \Phi' \\ a\Phi'' x_2 \\ a\Phi'' x_k \\ \vdots \end{bmatrix}' \hat{\Omega}_\beta \begin{bmatrix} a\Phi'' x_1 + \Phi' \\ a\Phi'' x_2 \\ a\Phi'' x_k \\ \vdots \end{bmatrix}$$

$$\sigma^2_{\mu_2} = \begin{bmatrix} b\Phi'' x_1 \\ b\Phi'' x_2 + \Phi' \\ b\Phi'' x_k \\ \vdots \end{bmatrix}' \hat{\Omega}_\beta \begin{bmatrix} b\Phi'' x_1 \\ b\Phi'' x_2 + \Phi' \\ b\Phi'' x_k \\ \vdots \end{bmatrix}$$

$$\sigma^2_{\mu_{12}} = \begin{bmatrix} (\Phi''\beta_{12} + ab\Phi''')x_1 + b\Phi'' \\ (\Phi''\beta_{12} + ab\Phi''')x_2 + a\Phi'' \\ (\Phi''\beta_{12} + ab\Phi''')x_k \\ \vdots \end{bmatrix}' \hat{\Omega}_\beta \begin{bmatrix} (\Phi''\beta_{12} + ab\Phi''')x_1 + b\Phi'' \\ (\Phi''\beta_{12} + ab\Phi''')x_2 + a\Phi'' \\ (\Phi''\beta_{12} + ab\Phi''')x_k \\ \vdots \end{bmatrix}$$

where

$$\begin{aligned} \Phi' &= \Phi'(I) = \frac{1}{\sqrt{2\pi}} e^{-\frac{I^2}{2}} \\ \Phi'' &= \Phi''(I) = -I\Phi' \\ \Phi''' &= \Phi'''(I) = (I^2 - 1)\Phi' \end{aligned}$$

Thus, the standard errors can be obtained by estimating the above expressions at the desired point like the mean of all the variables and taking the square root of them.